Acyclic Colorings and Triangulations of Weakly Chordal Graphs

Andrew Lyons

Computation Institute, The University of Chicago, and

Mathematics and Computer Science Division, Argonne National Laboratory

lyonsam@gmail.com

Abstract

An acyclic coloring of a graph is a proper vertex coloring without bichromatic cycles. We show that the acyclic colorings of any weakly chordal graph G correspond to the proper colorings of triangulations of G. As a consequence, we obtain polynomial-time algorithms for the acyclic coloring problem and the perfect phylogeny problem on the class of weakly chordal graphs. Our results also imply linear time algorithms for a number of graph classes contained within this class, such as permutation graphs and distance-hereditary graphs.

1 Introduction

We give polynomial-time algorithms for two coloring-related problems when restricted to the class of weakly chordal graphs.

An acyclic coloring is a proper (vertex) coloring with the additional restriction that there be no bichromatic cycles. The problem of finding an optimal acyclic coloring is NP-hard even when restricted to bipartite graphs [13]. We show that the acyclic colorings of a weakly chordal graph G are exactly the proper colorings of triangulations of G. This allows us to obtain an optimal acyclic coloring of G in the following way. We first find a triangulation H of G that is optimal with respect to treewidth, then find an optimal proper coloring of H. As H is chordal, the latter step requires only linear time [16]. Our algorithm for acyclic coloring on weakly chordal graphs works in this way. We are also able to use this idea to obtain linear time algorithms for the acyclic coloring problem on the distance-hereditary graphs, the permutation graphs, and other well-studied classes.

Our results also imply a polynomial-time algorithm for a related problem of independent interest. In the *colored graph triangulation* problem, a graph G is given along with a proper coloring ϕ, and we are to determine whether or not there exists some triangulation H of G for which ϕ is a proper coloring. In Sections 3 and 4, we describe a constructive, polynomial-time algorithm for this problem when restricted to weakly chordal graphs. The algorithm can be used to construct either a consistent triangulation or a corresponding tree decomposition.

2 Preliminaries

We consider finite, simple, undirected graphs that are assumed without loss of generality to be connected. If G is a graph, the vertex and edge sets of G are denoted by $V(G)$ and $E(G)$ (or simple V and E where there is no ambiguity). For $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S.

define path,cycle,chord.

A graph is *chordal* if it has no induced cycle on four or more vertices. A *triangulation*(chordal completion) of a graph G is a chordal graph H such that $V(H) = V(G)$ and $E(G) \subseteq E(H)$. The *clique number* of G, denoted $\omega(G)$, is size of the largest set of pairwise adjacent vertices in G.

2.1 Acyclic and proper colorings

A *proper vertex k-coloring* (or *proper coloring*) of a graph is a map $\phi : V(G) \to \{1, \dots, k\}$ such that that $xy \in E(G)$ implies $\phi(x) \neq \phi(y)$. The smallest k for which ϕ is a proper k-coloring of G is the *chromatic* *number* $\chi(G)$. An *acyclic coloring* of a graph G is a proper coloring such that the graph induced by the union of any two color classes contains no cycle. In other words, $G[A \cup B]$ is a forest for any two color classes A, B. As required by our applications, a solution of the ACYCLIC COLORING problem is an acyclic $\chi_a(G)$-coloring of G. We will only consider algorithms that are constructive in this sense.

In summary, we have the following.

Proposition 1 ([4, 15]). *For every chordal graph G, $\chi_a(G) = \chi(G) = \omega(G)$.* □

2.2 Triangulating colored graphs

Our results also apply to the *perfect phylogeny* problem, which has been shown to be polynomially equivalent to the following problem [12].

TRIANGULATING COLORED GRAPHS (TCG)
Instance: Graph G and a proper coloring ϕ of G.
Question: Does there exist a triangulation H of G such that ϕ is a proper coloring of H?

If the answer to this this question is yes, then we say that G is *ϕ-triangulatable* and H is a *ϕ-triangulation of G*.

Theorem 1 ([20, 2]). *There exists a polynomial-time algorithm that, given a graph G and a 3-coloring ϕ of G, determines whether G can be ϕ-triangulated.*

Not only is this problem NP-complete even when each color class consists of exactly two vertices, but it is not likely to be fixed-parameter tractable as it is complete for the W hierarchy [4]. We show that, for the weakly chordal graphs, this problem amounts to checking whether a given coloring of a graph is an acyclic coloring, which can be done in polynomial time. Moreover, we describe a polynomial-time algorithm that is *constructive*, in that, given G and ϕ, the algorithm finds a ϕ-triangulation of G if one exists.

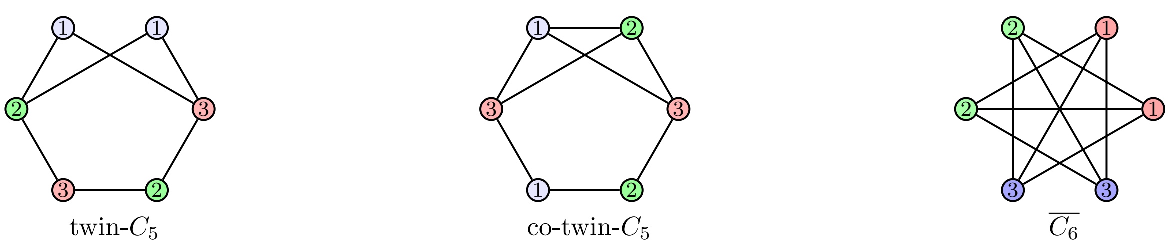


Figure 1: Three graphs, each given with an acyclic coloring ϕ that is not a proper coloring of any triangulation.

2.3 Weakly chordal graphs

A *hole* in a graph is an induced cycle on five or more vertices[1]. An *antihole* is the complement of a hole. A graph G is *weakly chordal* (also called *weakly triangulated*) if it contains no induced cycle or anticycle on five or more vertices. The class of weakly chordal graphs generalizes the chordal graphs while maintaining the property of being perfect [].

Though the bichromatic cycles prohibited by acyclic coloring are not required to be induced, it is easy to see that G contains a bichromatic cycle if and only if G contains an induced bichromatic cycle. The following statement, which follows from the fact that weakly chordal graphs are hole-free, will prove useful in limiting the types of cycles that we must consider.

Proposition 2. *Let uv be an edge in a weakly chordal graph G. If uv is part of a bichromatic cycle in G, then there exist vertices $t, w \in V(G)$ such that $\{t, u, v, w\}$ induces a bichromatic C_4 in G.* □

[1] Note that there is not a good consensus on the use of the term "hole". For instance, the class of even-hole-free graphs does not include graphs with induced cycles of length four. We follow the usage in [10].

Figure 2: Forbidden induced subgraphs for weakly chordal graphs.

3 Acyclic Coloring and Triangulation

Our first main result is the following.

Theorem 2. *If ϕ is a proper coloring of a weakly chordal graph G, then ϕ is an acyclic coloring of G if and only if ϕ is a proper coloring of some triangulation of G.*

We will prove Theorem 2 by showing that any weakly chordal graph given with an acyclic coloring can be triangulated without creating a bichromatic cycle.

Definition 1 (two-pair)**.** *A pair $\{x, y\}$ of distinct, non-adjacent vertices is a* two-pair *if every induced path from x to y consists of exactly two edges.*

Given a weakly chordal graph along with an acyclic coloring, our task is to show that G can be triangulated without creating a bichromatic cycle.

Theorem 3 ([18])**.** *If G is a weakly triangulated graph, then every induced subgraph of G that is not a clique contains a two-pair.*

3.1 Connecting two-pairs

Lemma 1 (key lemma)**.** *Let ϕ be an acyclic coloring of a graph G. If $\{x, y\}$ is a two-pair in G, then either $\phi(x) \neq \phi(y)$ or $\phi(u) \neq \phi(v)$ for all $u, v \in N(x) \cap N(y)$.*

Proof. If $\phi(x) = \phi(y)$ and there exist $u, v \in N(x) \cap N(y)$ such that $\phi(u) = \phi(v)$, then $xuyv$ is a bichromatic cycle in G. $\square$

We make use of this lemma in the following way. For a two-pair $\{x, y\}$ in G, we either We are careful we mantain the properties of G being weakly chordal and ϕ being an acyclic coloring of G.

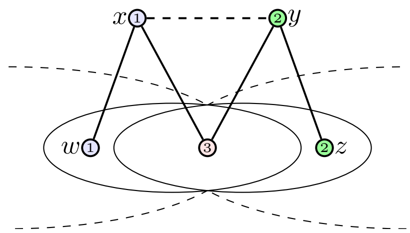
Let $G + xy$ denote the graph obtained by adding edge xy to G. The following lemma is due to Spinrad and Sritharan.

Lemma 2 ([24])**.** *If $\{x, y\}$ is a two-pair in a graph G, then G is weakly chordal if and only if $G + xy$ is weakly chordal.*

Lemma 3. *Let ϕ be an acyclic coloring of a graph G. If $\{x, y\}$ is a two-pair in G such that $\phi(x) \neq \phi(y)$, then ϕ is an acyclic coloring of $G + xy$.*

Proof. If $G + xy$ contains a bichromatic cycle $C = (x, c_1, \ldots, c_k, y)$, then by definition of two-pair we must have either $c_1 y \in E(G)$ or $xc_2 \in E(G)$. Either way, it follows that C cannot be bichromatic; a contradiction. $\square$

Lemma 4. *If $\{x, y\}$ is a two-pair in a weakly chordal graph G, then the graph G' obtained by turning $N(x) \cap N(y)$ into a clique is weakly chordal.*



Lemma **??**

Figure 3: (Lemma 3)

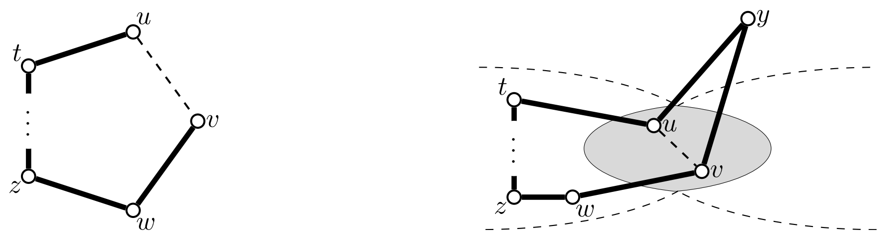


Figure 4: (proof of Lemma 4) The addition of edge uv cannot create a hole.

Proof. Suppose G' contains a hole. Any such hole C must contain exactly one edge uv where $u, v \in S$, thus C induces a path on at least five vertices in G. Let G_x and G_y denote the connected components of $G[V$§$-$ that contain x and y, respectively. Because the removal of S separates x and y, H must be disjoint from at least one of X or Y. Assume without loss of generality that H is disjoint from Y. It follows that $G[H \cup \{y\}]$ is a hole, which contradicts the fact that G is weakly chordal. Thus G' is hole-free.

Suppose G' contains an antihole A. We have already shown that G' contains no $C_5 = \overline{C_5}$, so A must have at least 6 vertices. Suppose $uv \in E' \setminus E$ is part of A. Then there exist distinct adjacent vertices $t, w \in V$ such that $tu, vw \notin E(G)$ and $vt, tw, wu \in E(G)$ Since neither t nor w can be contained in $N(x) \cap N(y)$ (which is now a clique), we may assume without loss of generality that $t, w \in X$ for some connected component X of $G[V \setminus S]$ such that $y \notin X$. This situation is depicted in Figure 5. It follows that $\{w, u, y, v, t\}$ induces a hole in G, which contradicts the fact that G is weakly chordal.

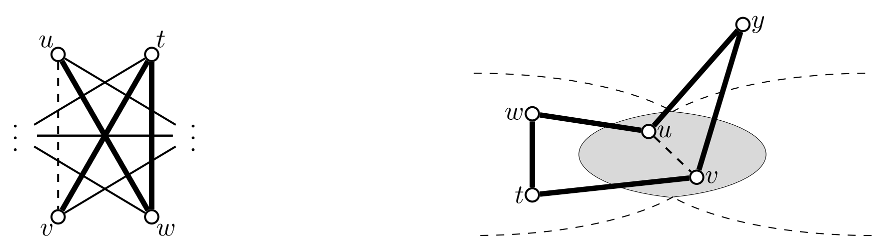


Figure 5: (proof of Lemma 4) The addition of edge uv cannot create an antihole.

We have shown that G' can contain neither a hole nor an antihole. It follows that G' is weakly chordal, which completes the proof of the lemma. □

Lemma 5. *Let $\{x, y\}$ be a two-pair in a weakly chordal graph G and let $S = N(x) \cap N(y)$. If ϕ is an acyclic coloring of G such that $\phi(u) \neq \phi(v)$ for all $u, v \in N(x) \cap N(y)$, then ϕ is an acyclic coloring of the graph G' obtained by turning $N(x) \cap N(y)$ into a clique.*

Proof. Suppose G' contains some bichromatic cycle $C \subseteq V$. If C also induces a cycle in G, then C cannot induce a bichromatic cycle in G'. It follows, then, that $|C \cap S| = \{u, v\}$ for distinct $u, v \in S$ such that

$uv \notin E(G)$ and $uv \in E(G')$, and thus C induces a path in G. Since S is an xy-separator, we may assume without loss of generality that $C - S \subseteq A$ for some connected component A of $G - S$ such that $y \notin A$. It follows, then, that $C \cup \{y\}$ induces a hole in G, which contradicts the fact that G is weakly chordal. Thus any set of vertices that induces an even cycle in G' also induces an even cycle in G, from which we may conclude that ϕ is an acyclic coloring of G'. ☐

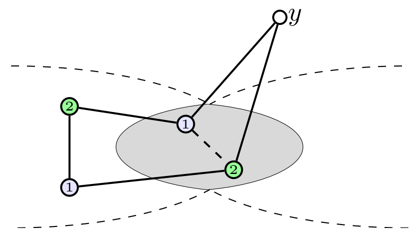


Figure 6: (proof of Lemma 5)

3.2 Clique separators

Definition 2. *Let G be a connected graph. A set $S \subset V$ is a* separator *if $G - S$ is disconnected. S is an* x-y-separator *if x and y are contained in distinct components of $G[V \setminus S]$. S is a* clique separator *if $G[S]$ is a clique.*

We will make use of the fact that $N(x) \cap N(y)$ is an x-xy separator if $\{x, y\}$ is a two-pair.

A separator S is a *clique separator* if $G[S]$ is a clique.

Lemma 6. *Let G be a graph with clique separator S and let ϕ be a proper coloring of G. G is ϕ-triangulatable if and only if $G[S \cup R]$ is ϕ-triangulatable for every connected component R of $G - S$.*

Proof. Since every $G[S \cup R]$ is an induced subgraph of G, every ϕ-triangulation of G is also a ϕ-triangulation of $G[S \cup R]$.

Now suppose G contains a chordless cycle C. We will show that C must be contained entirely in $G[S \cup R]$ for some connected component R of $G - S$. ... ☐

3.3 Main result

proof of Theorem 2. Clearly any proper coloring of a triangulation of G is an acyclic coloring of G.

We proceed with the other direction, which obviously holds if G is a clique. If G is not a clique, then G contains a two-pair $\{x, y\}$ by Theorem 3. Suppose ϕ is an acyclic coloring of G. Then either $\phi(x) \neq \phi(y)$ or no two vertices in $N(x) \cap N(y)$ share a color, as we would otherwise have a bichromatic cycle. We now construct a new graph G' as follows. If $\phi(x) \neq \phi(y)$, then $G' = G + xy$. If no two vertices in $N(x) \cap N(y)$ share a color, then G' is obtained by adding edges to make $N(x) \cap N(y)$ a clique. In either case, the G' is a weakly chordal graph for which ϕ is an acyclic coloring. Continuing inductively in this fashion, we invariably arrive (in a finite number of steps) at a weakly chordal graph for which ϕ is an acyclic coloring, and which does not contain a two-pair. Such a graph must be a clique, which completed the proof of the theorem. ☐

Corollary 4. *If G is a weakly chordal graph, then $\chi_a(G) = \mathrm{tw}(G) + 1$.*

4 Algorithms

4.1 Treewidth

The *treewidth* of a graph G, denoted $\mathrm{tw}(G)$, is the minimum value of $\omega(G^+) - 1$ over all triangulations G^+ of G. For our purposes, a solution of TREEWIDTH will consist of a triangulation H of G that satisfies

$\chi(H) = \mathrm{tw}(G) + 1$. TREEWIDTHis NP-complete in general, though it can be determined in linear time whether G has treewidth at most k for fixed k [3].

There has also been work on exact algorithms for treewidth [14].

In 1999, Bouchitté and Todinca [8] gave an $O(n^6)$ time for computing the treewidth of a weakly chordal graph. Their algorithm is constructive – it builds a treewidth-optimal triangulation.

One technique for designing fixed-parameter algorithms [1]...In contrast to most uses of treewidth, we do not rely on this parameter being bounded.

We will use the fact that TREEWIDTHcan be solved in polynomial time for weakly chordal graphs.

Generally, the difference between $\mathrm{tw}(G)$ and $\chi_a(G)$ can be large. The planar graphs, for example, have acyclic chromatic number no greater than five [6, 7], while the treewidth of planar graphs is unbounded. In this paper, we consider a class of graphs for which $\chi_a(G) = \mathrm{tw}(G) + 1$.

4.2 Algorithms for TRIANGULATING COLORED GRAPHS

Given a weakly chordal graph G that is properly colored by ϕ, we can test in polynomial time whether ϕ is an acyclic coloring of G. In order to construct a coresponding triangulation, we may apply the process described in the proof of Theorem **??**.

4.3 Algorithms for ACYCLIC COLORING

The best known algorithms for determining the treewidth of a weakly chordal graph use the fact that graphs in this class have a polynomial number of minimal separators [9].

Theorem 5. *If $\mathcal{C}$ is a subclass of the weakly chordal graphs for which* TREEWIDTH *can be solved in $f_{\mathcal{C}}(n, m)$ time for every $G \in \mathcal{C}$, then* ACYCLIC COLORING *can be solved in $O(f_{\mathcal{C}}(n, m) + n + m)$ time for every $G \in \mathcal{C}$.*

Proof. The proof follows from Theorem 2 and the fact that a proper coloring of a chordal graph can be found in $O(n + m)$ time. □

Corollary 6. *Given a weakly chordal graph G, an optimal acyclic coloring of G can be found in*

Biconvex graphs. Linear time [23].

P_4-sparse graphs.

Permutation graphs. First there was an $O(nk)$ algorithm, where k is the treewidth [5]. Then Meister described an $O(n + m)$ algorithm [22].

Distance-Hereditary graphs.

4.4 The perfect phylogeny problem

5 Graphs for which acyclic colorings can be triangulated

Theorem 7 ([20])**.** *Let G be a simple cycle on k vertices with a proper coloring ϕ. Then G can be ϕ-triangulated if and only if ϕ is an acyclic coloring of G.*

Theorem 8. *Let G be an antihole on k vertices such that $k \neq 6$. Then ϕ is an acyclic coloring of G if and only if ϕ is a proper coloring of some triangulation of G.*

Proof. The proof is trivial for $k \leq 4$. Since $C_5 = \overline{C_5}$, the case for $k = 5$ is addressed by Theorem 7. Now assume $k \geq 7$. □

$\overline{C_6}$ a triangulation

Figure 7: There is a unique (up to isomorphism) minimal triangulation of $\overline{C_6}$. This is true of any antihole.

6 Conclusions and open problems

Problem 1. *Find a $O(n)$ time algorithm for coloring a Ptolemaic graph that is given in the form of a split tree. (This would mean an $O(n)$ time algorithm for the acyclic coloring problem on distance-hereditary graphs that are given in the form of a split tree.)*

Problem 2. *Find a $o(n^{??4??})$ time algorithm for the acyclic coloring problem on weakly chordal graphs.*

Acknowledgments

Kathie Cameron, Jeremy Spinrad. This work was supported in part by U.S. Department of Energy, under Contract DE-AC02-06CH11357.

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APPENDIX

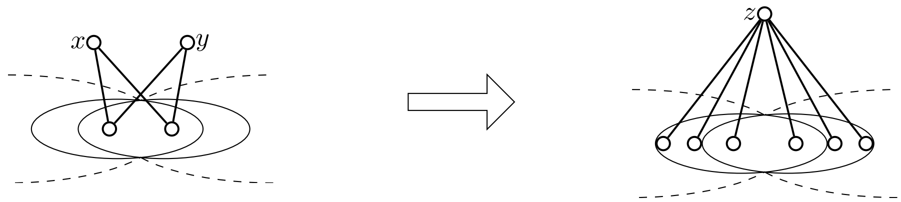

Figure 8: Two-pair $\{x, y\}$.

Let $G(xy \to z)$ denote the graph obtained by identifying x and y as a new vertex z with $N(z) = N(x) \cup N(y)$.

Lemma 7 (Identification Lemma [18])**.** *If $\{x, y\}$ is a two-pair in a weakly chordal graph G, then $G(xy \to z)$ is weakly chordal.*

Lemma 8. *Let $\{x, y\}$ be a two-pair in a weakly chordal graph G. If ϕ is an acyclic coloring of G such that $\phi(x) = \phi(y)$, then ϕ is an acyclic coloring of $G(xy \to z)$, where $\phi(z) = \phi(x)$.*

Proof. Suppose ϕ causes a bichromatic cycle C in $G(xy \to z)$. Since G contains no bichromatic cycles, we must have $C = (z, c_1, c_2, \ldots, c_k)$, where $k \geq 3$. $\square$